李昱老師機率數統第 2 次模考 (112 年考生) (滿分 150 分)
機率論: CH1(全), CH2(全), CH3(全), CH4(全)
考試時間: 100 分鐘 [可使用計算機]
[請保留有效位數至少四位以上, 有效位數意指從第一個不是零的位數開始往後計算。]
[計算證明題請將所有過程盡量寫清楚，不完整的內容可能會被扣分。]

1. An insurance company finds that 0.005 percent of the population die from a certain kind of accident each year. What is the probability that the company must pay off on more than 3 of 10,000 insured risks against such accidents in a given year? (10%)

[交大統研]

2. Let $\mathcal{B}$ stand for the Beta function. The distribution, for $x \in [-1, 1]$, given by
$$f(x; n) = \frac{1}{\mathcal{B}(1/2, (n-2)/2)} \left(1 - x^2\right)^{(n-4)/2}$$
is called the r distribution. Find the mean and the variance of this distribution. (15%)

[交大統研]

3. If X has a uniform distribution over the interval $(-\pi/2, \pi/2)$, find the distribution of $Y = \tan X$. (15%)

[交大統研]

4. Assume $X_1, ..., X_n$ are iid $\mathcal{U}(0, 1)$. Let $T = -2\sum_{i=1}^{n} \ln(X_i)$ Please derive the pdf of T. (15%)

[中山應數]

5. Assume Y is a random variable with a Poisson distribution with parameter λ. However, λ is a random variable with pdf
$$f(\lambda) = e^{-\lambda} I_{(\lambda>0)}$$
where $I(\cdot)$ is the indicator function. Please calculate $\mathrm{Var}(Y)$. (15%)

[中山應數]

6. Let the joint probability density function of (X, Y) be
$$f(x, y) = 1, 0 < x < 1, x < y < x + 1.$$
   (a) Find the marginal probability density functions of X and Y. (10%)
   (b) Find the covariance of X and Y. (10%)

[中央統研]

7. Let $X_1 < X_2 < ... < X_n$ be the order statistics of n independent observations from a $\mathcal{U}(0, 1)$ distribution and let Y_n be $Y_n = X_n - X_1$. Please calculate the expectation of Y_n as $n \to \infty$. (15%)

[中央統研]

8. Let $Y = (Y_1, \cdots, Y_c) \sim \text{Multinomial}(n; \pi_1, \cdots, \pi_c)$ represent a multinomial distribution with pdf

$$P(Y_1 = y_1, \cdots, Y_c = y_c) = \frac{n!}{y_1! \cdots y_c!} \prod_{i=1}^{c} \pi_i^{y_i},$$

subject to $\sum_{i=1}^{c} y_i = n$ and $\sum_{i=1}^{c} \pi_i = 1$. Obtain

(a) Find the moment generating function of Y, which is $M_{Y_1 \cdots Y_c}(t_1, \cdots, t_c) = E(e^{t_1 Y_1 + \cdots + t_c Y_c})$. (15%)

(b) Show that $Y_i | Y_k = y_k \sim \text{Binomial}\left(n - y_k, \frac{\pi_i}{1 - \pi_k}\right)$. (15%)

(c) Show that $\text{Cov}(Y_i, Y_k) = -n\pi_i\pi_k$. (15%)

[中興統研]

[bonus] Let X and Y be independent standard normal random variables. Find the moment generating function of XY. (10%)

[交大統研]

1. An insurance company finds that 0.005 percent of the population die from a certain kind of accident each year. What is the probability that the company must pay off on more than 3 of 10,000 insured risks against such accidents in a given year? (10%)

[交大統研]

sol: 令 X 表示 10,000 件事件中死亡人數

依題意可知 $X \sim \text{Bin}(n = 10,000, p = 0.00005) \xrightarrow{a} \text{Poi}(\lambda = np = 0.5)$

則所求為 $P(X > 3) = 1 - P(X \leqslant 3) = 1 - \sum_{x=0}^{3} \frac{e^{-0.5}(0.5)^x}{x!} = 0.001752$

□

2. Let $\mathcal{B}$ stand for the Beta function. The distribution, for $x \in [-1, 1]$, given by

$$f(x; n) = \frac{1}{\mathcal{B}\left(\frac{1}{2}, \frac{n-2}{2}\right)} \left(1 - x^2\right)^{(n-4)/2}$$

is called the r distribution. Find the mean and the variance of this distribution. (15%)

[交大統研]

sol: $E(X) = \int_{-1}^{1} x \frac{1}{\mathcal{B}\left(\frac{1}{2}, \frac{n-2}{2}\right)} \left(1 - x^2\right)^{(n-4)/2} dx$

由於 $\frac{1}{\mathcal{B}\left(\frac{1}{2}, \frac{n-2}{2}\right)} \left(1 - x^2\right)^{(n-4)/2}$ 為一個偶函數

故可知 $x \frac{1}{\mathcal{B}\left(\frac{1}{2}, \frac{n-2}{2}\right)} \left(1 - x^2\right)^{(n-4)/2}$ 為一個奇函數

則 $E(X) = \int_{-1}^{1} x \frac{1}{\mathcal{B}\left(\frac{1}{2}, \frac{n-2}{2}\right)} \left(1 - x^2\right)^{(n-4)/2} dx = 0$

$\Longrightarrow \text{Var}(X) = E\left(X^2\right) = \int_{-1}^{1} x^2 \frac{1}{\mathcal{B}\left(\frac{1}{2}, \frac{n-2}{2}\right)} \left(1 - x^2\right)^{(n-4)/2} dx$

又由於 $x^2 \frac{1}{\mathcal{B}\left(\frac{1}{2}, \frac{n-2}{2}\right)} \left(1 - x^2\right)^{(n-4)/2}$ 為一偶函數

故我們有 $\text{Var}(X) = E(X^2) = \int_{-1}^{1} x^2 \frac{1}{\mathcal{B}\left(\frac{1}{2}, \frac{n-2}{2}\right)} \left(1 - x^2\right)^{(n-4)/2} dx$

$$= 2\int_{0}^{1} x^2 \frac{1}{\mathcal{B}\left(\frac{1}{2}, \frac{n-2}{2}\right)} \left(1 - x^2\right)^{(n-4)/2} dx \qquad (\text{令 } y = x^2)$$

$$= 2\int_{0}^{1} y \frac{1}{\mathcal{B}\left(\frac{1}{2}, \frac{n-2}{2}\right)} (1 - y)^{\frac{n-4}{2}} \frac{1}{2\sqrt{y}} dy$$

$$= \frac{1}{\mathcal{B}\left(\frac{1}{2}, \frac{n-2}{2}\right)} \int_0^1 y^{\frac{3}{2}-1}(1-y)^{\frac{n-2}{2}-1}\, dy$$

$$= \frac{1}{\mathcal{B}\left(\frac{1}{2}, \frac{n-2}{2}\right)} \times \mathcal{B}\left(\frac{3}{2}, \frac{n-2}{2}\right) = \frac{\Gamma\left(\frac{1}{2}+\frac{n-2}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n-2}{2}\right)} \times \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{n-2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \times \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\frac{n-1}{2}\Gamma\left(\frac{n-1}{2}\right)} = \frac{1}{n-1}$$

□

3. If X has a uniform distribution over the interval $(-\pi/2, \pi/2)$, find the distribution of $Y = \tan X$. (15%)

[交大統研]

sol: 依題意可知 $X \sim f_X(x) = \frac{1}{\pi},\ -\frac{\pi}{2} < x < \frac{\pi}{2}$

則由 Jacobian 法, 我們有 $Y = \tan X \iff X = \tan^{-1} Y \iff \mathcal{J} = \frac{dx}{dy} = \frac{1}{1+y^2}$

$\implies f_Y(y) = f_X\left(\tan^{-1} y\right)|\mathcal{J}| = \frac{1}{\pi(1+y^2)},\ y \in \mathbb{R}$

□

4. Assume $X_1, ..., X_n$ are iid $\mathcal{U}(0,1)$. Let $T = -2\sum_{i=1}^{n} \ln(X_i)$ Please derive the pdf of T. (15%)

[中山應數]

sol: 由 Jacobian 法令 $Y_i = -2\ln X_i \implies X_i = e^{-\frac{Y_i}{2}} \implies \mathcal{J} = \frac{dx}{dy} = -\frac{1}{2}e^{-\frac{y}{2}}$

$\implies f_{Y_i}(y) = f_{X_i}\left(e^{-\frac{y}{2}}\right)|\mathcal{J}| = \frac{1}{2}e^{-\frac{y}{2}},\ y > 0$

此即 $Y_i \overset{iid}{\sim} \text{Exp}(\beta = 2) \sim \text{Gamma}(\alpha = 1, \beta = 2) \sim \chi^2(\nu = 2)$

又令 $T = -2\sum_{i=1}^{n} \ln X_i = \sum_{i=1}^{n} Y_i \sim \text{Gamma}(\alpha = n, \beta = 2) \sim \chi^2(\nu = 2n)$

可知 $f_T(t) = \frac{1}{\Gamma(n)2^n} t^{n-1} e^{-\frac{t}{2}},\ t > 0$

□

5. Assume Y is a random variable with a Poisson distribution with parameter λ. However, λ is a random variable with pdf

$$f(\lambda) = e^{-\lambda} I_{(\lambda > 0)}$$

where $I(\cdot)$ is the indicator function. Please calculate $\text{Var}(Y)$. (15%)

[中山應數]

sol: 由題意可知 $Y|\lambda \sim \text{Poi}(\lambda),\ \lambda \sim \text{Exp}(\beta = 1)$

則由變異數分解定理可知 $\text{Var}(Y) = E\left[\text{Var}(Y|\lambda)\right] + \text{Var}\left[E(Y|\lambda)\right] = E(\lambda) + \text{Var}(\lambda) = 1 + 1 = 2$

□

6. Let the joint probability density function of (X, Y) be

$$f(x, y) = 1, 0 < x < 1, x < y < x + 1.$$

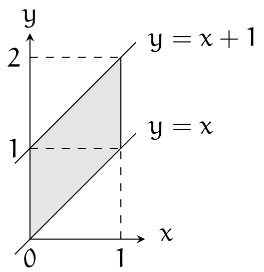
(a) Find the marginal probability density functions of X and Y. (10%)

(b) Find the covariance of X and Y. (10%)

[中央統研]

sol: (a) 本題所敘述的值域範圍如右圖所示

故我們可由值域範圍決定邊際 pdf 的積分範圍



$$f_X(x) = \int_x^{x+1} 1\,dy = 1,\ 0 < x < 1$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\,dx = \begin{cases} \int_0^y 1\,dx = y & , 0 < y < 1 \\ \int_{y-1}^1 1\,dx = 2 - y & , 1 < y < 2 \end{cases}$$

(b) $$E(XY) = \int_0^1 \int_x^{x+1} xy \times 1\,dy\,dx = \int_0^1 x \frac{1}{2}\left[(x+1)^2 - x^2\right] dx$$

$$= \frac{1}{2}\int_0^1 \left(2x^2 + x\right) dx = \frac{1}{2}\left(\frac{2}{3} + \frac{1}{2}\right) = \frac{7}{12}$$

$$E(X) = \int_0^1 x \times 1\,dx = \frac{1}{2},\ \ E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy = 1$$

$$\Longrightarrow \mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y) = \frac{7}{12} - \frac{1}{2} \times 1 = \frac{1}{12}$$

□

7. Let $X_1 < X_2 < \ldots < X_n$ be the order statistics of n independent observations from a $\mathcal{U}(0, 1)$ distribution and let Y_n be $Y_n = X_n - X_1$. Please calculate the expectation of Y_n as $n \to \infty$. (15%)

[中央統研]

sol: 由順序統計量之公式, 我們有 $f_{X_k}(x) = \dfrac{n!}{(k-1)!(n-k)!} x^{k-1}(1-x)^{n-k},\ 0 < x < 1$

此即 $X_k \sim \mathrm{Beta}(k, n+1-k),\ k = 1, 2, \ldots, n$

則所求為 $E(Y_n) = E(X_n) - E(X_1) = \dfrac{n}{n+1} - \dfrac{1}{n+1} = \dfrac{n-1}{n+1}$

$$\Longrightarrow \lim_{n\to\infty} E(Y_n) = \lim_{n\to\infty} \frac{n-1}{n+1} = 1$$

□

筆記 本題雖然可以使用 X_n 與 X_1 的聯合分配處理, 但由於題目只要求期望值之差, 故可以直接使用邊際分配來計算較為簡單。

8. Let $Y = (Y_1, \cdots, Y_c) \sim \text{Multinomial}(n; \pi_1, \cdots, \pi_c)$ represent a multinomial distribution with pdf

$$P(Y_1 = y_1, \cdots, Y_c = y_c) = \frac{n!}{y_1! \cdots y_c!} \prod_{i=1}^{c} \pi_i^{y_i},$$

subject to $\sum_{i=1}^{c} y_i = n$ and $\sum_{i=1}^{c} \pi_i = 1$. Obtain

(a) Find the moment generating function of Y, which is $M_{Y_1 \cdots Y_c}(t_1, \cdots, t_c) = E(e^{t_1 Y_1 + \ldots + t_c Y_c})$. (15%)

(b) Show that $Y_i | Y_k = y_k \sim \text{Binomial}\left(n - y_k, \dfrac{\pi_i}{1 - \pi_k}\right)$. (15%)

(c) Show that $\text{Cov}(Y_i, Y_k) = -n\pi_i\pi_k$. (15%)

[中興統研]

sol: (a) $M_{Y_1 \cdots Y_c}(t_1, \ldots, t_c) = E\big(\exp\{t_1 Y_1 + \ldots + t_c Y_c\}\big)$

$$= \sum \cdots \sum \exp\{t_1 y_1 + \ldots + t_c y_c\} \frac{n!}{\prod_{i=1}^{c} y_i!} \prod_{i=1}^{c} \pi_i^{y_i}$$

$$= \sum \cdots \sum \frac{n!}{\prod_{i=1}^{c} y_i!} \prod_{i=1}^{c} (\pi_i e^{t_i})^{y_i} = \left(\sum_{i=1}^{c} \pi_i e^{t_i}\right)^n, \quad t_i \in \mathbb{R}, i = 1, \ldots, c$$

(b) $M_{Y_i Y_k}(t_i, t_k) = M_{Y_1 \cdots Y_c}(0, \ldots, 0, t_i, 0, \ldots, 0, t_k, 0, \ldots, 0)$

$$= (\pi_i e^{t_i} + \pi_k e^{t_k} + 1 - \pi_i - \pi_k)^n, \quad t_i, t_k \in \mathbb{R}$$

由 mgf 的唯一性可知 $(Y_i, Y_k) \sim \text{trinomial}(n, \pi_i, \pi_k)$

又進一步有 $M_{Y_k}(t_k) = M_{Y_i Y_k}(0, t_k) = (\pi_k e^{t_k} + 1 - \pi_k)^n$, $t_i \in \mathbb{R}$, 可知 $Y_k \sim \text{Bin}(n, \pi_k)$

$$\Longrightarrow f_{Y_i|Y_k}(y_i|y_k) = \frac{f_{Y_i Y_k}(y_i, y_k)}{f_{Y_k}(y_k)} = \frac{\frac{n!}{y_i! y_k! (n - y_i - y_k)!} \pi_i^{y_i} \pi_k^{y_k} (1 - \pi_i - \pi_k)^{n - y_i - y_k}}{\frac{n!}{y_k! (n - y_k)!} \pi_k^{y_k} (1 - \pi_k)^{n - y_k}}$$

$$= \frac{(n - y_k)!}{y_i! (n - y_i - y_k)!} \frac{\pi_i^{y_i} (1 - \pi_i - \pi_k)^{n - y_i - y_k}}{(1 - \pi_k)^{n - y_k}}$$

$$= \binom{n - y_k}{y_i} \left(\frac{\pi_i}{1 - \pi_k}\right)^{y_i} \left(1 - \frac{\pi_i}{1 - \pi_k}\right)^{n - y_i - y_k}, \ y_i = 0, 1, \ldots, n - y_k$$

此即說明 $Y_i | Y_k = y_k \sim \text{Bin}\left(n - y_k, \dfrac{\pi_i}{1 - \pi_k}\right)$

(c) 由前述結果可知 $E(Y_i | Y_k = y_k) = (n - y_k)\left(\dfrac{\pi_i}{1 - \pi_k}\right) = \dfrac{-\pi_i}{1 - \pi_k} y_k + \dfrac{n\pi_i}{1 - \pi_k}$

同理可知 $E(Y_k | Y_i = y_i) = (n - y_i)\left(\dfrac{\pi_k}{1 - \pi_i}\right) = \dfrac{-\pi_k}{1 - \pi_i} y_i + \dfrac{n\pi_k}{1 - \pi_i}$

則由母體線性迴歸方程式可知 $\rho_{ik} = -\sqrt{\left(\dfrac{-\pi_i}{1 - \pi_k}\right)\left(\dfrac{-\pi_k}{1 - \pi_i}\right)}$

$$\Longrightarrow \text{Cov}(Y_i, Y_k) = \rho_{ik}\sigma_i\sigma_j = -\sqrt{\left(\frac{-\pi_i}{1-\pi_k}\right)\left(\frac{-\pi_k}{1-\pi_i}\right)}\sqrt{n\pi_i(1-\pi_i)}\sqrt{n\pi_k(1-\pi_k)}$$

$$= -n\pi_i\pi_k$$

□

[bonus] Let X and Y be independent standard normal random variables. Find the moment generating function of XY. (10%)

[交大統研]

sol: 令 $T = XY$, 則可知 $M_T(t) = E(e^{tT}) = E(e^{tXY}) = E\Big[E(e^{tXY}|X)\Big]$

又其中, $E(e^{tXY}|X = x) = E(e^{txY}) = M_Y(tx) = \exp\left\{\frac{t^2x^2}{2}\right\}$

故可知 $M_T(t) = E(e^{tXY}) = E\Big[E(e^{tXY}|X)\Big] = E\left(\exp\left\{\frac{t^2X^2}{2}\right\}\right)$

$$= \int_{-\infty}^{\infty} \exp\left\{\frac{t^2x^2}{2}\right\} \times \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(1-t^2)x^2}{2}\right\} dx$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{2\pi\frac{1}{1-t^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\frac{1}{1-t^2}}} \exp\left\{-\frac{x^2}{2\frac{1}{1-t^2}}\right\} dx$$

$$= \frac{1}{\sqrt{2\pi}}\sqrt{2\pi\frac{1}{1-t^2}} \times 1 = \sqrt{\frac{1}{1-t^2}},\ -1 < t < 1$$

□